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BIMODAL SOLUTIONS IN EIGENVALUE OPTIMIZATION PROBLEMS*

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The problem of maximizing the minimum eigenvalue of a selfadjoint operator is examined. An isoperimetric condition is imposed on the control variable. This problem has interesting applications in the optimal design of structures. In papers on the optimization of the critical stability parameters and the frequencies of the natural oscillations of elastic systems /1—12/ it was shown that in a number of cases the optimal solutions are characterized by two or more forms of loss of stability or natural oscillations. In the case of conservative systems described by selfadjoint equations this signifies multiplicities of eigenvalues, i.e,, of critical loads, under which loss of stability or of natural oscillation frequencies occurs. The necessary conditions for an extremum are obtained in the case when the optimal solution is characterized by a double eigenvalue. These conditions have a constructive character and can be used for the numerical and analytical solution of optimization problems. Both discrete and continuous cases of the specification of the original system are analyzed. Examples are given.

1. The discrete case. Consider the eigenvalue problem

$$A\,[h]\,u = \lambda u \tag{1.1}$$

Here $A\,[h]$ is an $m\times m$-matrix with coefficients $a_{ij}\,(h)\,(i,\,j=1,\ldots,m)$ depending smoothly on the components of the vector h of dimensions n, u is an m-dimensional vector, and λ is an eigenvalue.

We formulate the optimization problem as follows: find a parameter vector $h=(h_1,\ h_2,\ldots,h_n)$ for which the minimum eigenvalue λ will be a maximum under the condition

$$V\,(h) = V_0 \tag{1.2}$$

where V is a scalar function and V_0 is a given constant. It is assumed that V is differentiable with respect to the variables $h_i\ (i=1,2,\ldots,n)$. We assume that a parameter vector $h_0=(h_1^\circ,h_2^\circ,\ldots,h_n^\circ)$ exists for which problem (1.1) has a double eigenvalue which is the smallest of all the eigenvalues. The eigenvectors corresponding to the multiple eigenvalue λ_0 are assumed to be orthogonal and normalized, and are denoted by u_1 and u_2. Any eigenvector u_0 corresponding to the double λ_0 can be represented as a linear combination of eigenvalues u_1 and u_2, with coefficients γ_1 and γ_2

$$u_0 = \gamma_1 u_1 + \gamma_2 u_2 \tag{1.3}$$

We give an increment εk, to the parameter vector h, where k is an n-dimensional vector and ε is a small number, and we find the increments of the double eigenvalue and the eigenvectors /13/

$$u = \gamma_1 u_1 + \gamma_2 u_2 + \varepsilon v + o\,(\varepsilon),\ \ \lambda = \lambda_0 + \varepsilon\mu + o\,(\varepsilon) \tag{1.4}$$

where γ_1,γ_2,μ,v are quantitites to be determined. Substituting expansions (1.4) into (1.1) and collecting terms of the order of ε, we obtain

$$A_1\,[h_0,\,k]u_0 + A\,[h_0]\,v = \lambda_0 v + \mu u_0 \tag{1.5}$$

where u_0 is defined by (1.3), and $A_1\,[h_0,\,k]$ is a symmetric $m\times m$-matrix with coefficients $(\nabla a_{ij},\,k)$, $i,\,j=1,\,2,\ldots m$, where

$$\nabla a_{ij} = \left(\frac{\partial a_{ij}}{\partial h_1},\,\frac{\partial a_{ij}}{\partial h_2},\,\ldots,\,\frac{\partial a_{ij}}{\partial h_n}\right),\quad k=(k_1,\,k_2,\,\ldots,\,k_n)$$

The partial derivatives of the coefficients of the matrix A are computed for $h=h_0$.

By scalar multiplication of Eq.(1.5) in succession by the vectors u_1 and u_2 and using the equation $A\,[h_0]u_i=\lambda_0 u_i$, $i=1,2$, as well as the orthogonality of the vectors u_1 and u_2, we obtain linear homogeneous equations in the unknown coefficients γ_1 and γ_2

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$$\gamma_1(\alpha_{11}-\mu)+\gamma_2\alpha_{12}=0,\ \gamma_1\alpha_{21}+\gamma_2(\alpha_{22}-\mu)=0$$
$$\alpha_{ij}=(A_1[h_0,k]u_i,u_j);\ i,j=1,2$$

For non-zero γ_1 and γ_2 to exist it is necessary and sufficient that the determinant of this system equal zero. This yields a quadratic equation in μ

$$\mu^2-(\alpha_{11}+\alpha_{22})\mu+(\alpha_{11}\alpha_{22}-\alpha_{12}^2)=0 \tag{1.6}$$

In view of the symmetry of the matrix $A_1[h,k]$ we have $\alpha_{12}=\alpha_{21}$. Hence it follows that the roots of Eq.(1.6) are real.

It is convenient to introduce the n-dimensional vectors

$$f_1=\sum_{ij=1}^{m}u_i^1u_j^1\nabla a_{ij},\quad f_2=\sum_{ij=1}^{m}u_i^2u_j^2\nabla a_{ij},\qquad f_3=\sum_{ij=1}^{m}u_i^1u_j^2\nabla a_{ij} \tag{1.7}$$

where u_i^1, u_i^2 $(i=1,2,\ldots,m)$ are the coordinates of the eigenvectors u_1, u_2. With due regard to the notation introduced we obtain $\alpha_{11}=(f_1,k)$, $\alpha_{12}=\alpha_{21}=(f_3,k)$, $\alpha_{22}=(f_2,k)$. Equation (1.6) has the solutions

$$\mu_{1,2}=\frac{(f_1+f_2,k)}{2}\pm\left[\frac{(f_1+f_2,k)^2}{4}+(f_3,k)^2-(f_1,k)(f_2,k)\right]^{1/2} \tag{1.8}$$

Thus, in accordance with formulas (1.7) we can compute the vectors f_1, f_2, f_3 from the matrix A and the eigenvectors μ_1 and μ_2, and then, by specifying the variation vector k, from (1.8) we obtain the increments μ_1 and μ_2 of the eigenvalue. Note that when k changes sign the eigenvalue increments change their signs as well. It is important to distinguish the cases when the quantitites u_1 and u_2 have the same or different signs which, according to Viet's theorem $\mu_1\mu_2=(f_1,k)(f_2,k)-(f_3,k)^2$, are determined by the sign of the quadratic form in the components of the vector k.

By varying the isoperimetric condition (1.2) and equating to zero the linear part of the increment of the functional V with respect to ε, we obtain

$$(f_0,k)=0,\ f_0=\nabla V \tag{1.9}$$

The latter signifies that the variation of the parameter vector k must be orthogonal to the vector f_0. We will formulate the first statement.

1°. If the vectors f_0, f_1, f_2, f_3 are linearly independent $(n\geqslant 4)$, then a refining variation k of the parameter vector h_0 exists for which each of the perturbed eigenvalues is larger than the original eigenvalue λ_0, $\mu_1>0$, $\mu_2>0$.

Proof. Consider the quadratic form

$$L(k)=(f_3,k)^2-(f_1,k)(f_2,k) \tag{1.10}$$

If for some k the form (1.10) has negative values, then both roots μ_1 and μ_2 will have the same sign. If, in addition, $(f_3,k)=0$, then according to (1.8), $\mu_1=(f_1,k)$ and $\mu_2=(f_2,k)$.

Let us show that a non-zero vector k always exists, which is a solution of the system of linear equations

$$(f_0,k)=0,\ (f_3,k)=0,\ (f_1,k)=p_1>0,\ (f_2,k)=p_2>0 \tag{1.11}$$

Indeed, by virtue of the assumption that the vectors f_0, f_1, f_2, f_3 are linearly independent, the rank of this system's matrix equals four, which signifies the existence of non-zero vectors k satisfying the system of linear equations (1.11). The latter proves the possibility of constructing a refining variation in the case considered.

We will now study the case when the system of vectors f_0, f_1, f_2, f_3 is linearly dependent. Let us consider the linear subspace of all vectors spanned by the vectors f_0, f_1, f_2, f_3 and in this subspace choose an orthogonal basis f_0, e_1, e_2 (the dimensions of the subspace are no higher than three). We assume that $(f_0,f_0)=(e_1,e_1)=(e_2,e_2)\neq 0$. We expand the vectors f_1, f_2, f_3 with respect to the basis vectors

$$f_1=b_0f_0+b_1e_1+b_2e_2,\ f_2=c_0f_0+c_1e_1+c_2e_2$$
$$f_3=d_0f_0+d_1e_1+d_2e_2$$

Here b_i, c_i, d_i $(i=0,1,2)$ are the coordinates of the vectors f_1, f_2, f_3 in the basis chosen.

We complete the basis f_0, e_1, e_2 up to an orthogonal basis for the whole space, having added on to it the vectors $e_3, e_4, \ldots, e_{n-1}$. By virtue of condition (1.9) we represent the vector $k=l_1e_1+l_2e_2+\ldots+l_{n-1}e_{n-1}$ with arbitrary constants l_i $(i=1,2,\ldots,n-1)$.

Substituting the resultant expansions into (1.10), we obtain

$$L(k)=Dl_1^2+2Bl_1l_2+Cl_2^2 \tag{1.12}$$
$$D=d_1^2-b_1c_1,\ B=d_1d_2-1/2\ (b_1c_2+b_2c_1)$$
$$C=d_2^2-b_2c_2$$

We will formulate the next statement.

2°. If the parameter vector h_0 yields an extremum of the optimization problem with a double eigenvalue λ_0, then it is necessary that: 1) the system of vectors f_0, f_1, f_2, f_3 be linearly dependent and 2) the quadratic form (1.10) should admit of only non-negative values for any admissible non-zero vectors k, which is equivalent to the conditions $D \geqslant 0, DC - B^2 \geqslant 0$. where D, B, C are defined by Eqs. (1.12).

Proof. The necessity of the first condition follows from statement 1°. Let us prove the necessity of the second condition. Suppose that the vector h_0 yields an extremum of the optimization problem with a double eigenvalue λ_0. If form (1.10) admits of negative values for some vector k, then the roots μ_1, μ_2 will have the same sign. By changing, if necessary, the sign of the vector k we can arrange for both roots to be strictly positive, which indicates the possibility of choosing a refining variation which splits the multiple eigenvalue λ_0. Consequently, for an extremum it is necessary that form (1.10) be non-negative for all non-zero k. Using representation (1.12), from this we obtain $D \geqslant 0, DC - B^2 \geqslant 0$.

Note that form (1.10) can take zero values as well, as follows from the existence of non-zero vectors k orthogonal to the subspace formed by the vectors f_0, f_1, f_2, f_3.

The conditions imposed on the coefficients D, B, C, obtained above, can be constructed directly by using the linear dependence of the vectors f_0, f_1, f_2, f_3. To do this we will write the linear combination $\delta_0 f_0 + \delta_1 f_1 + \delta_2 f_2 + \delta_3 f_3 = 0$, where the coefficients δ_i do not all equal zero simultaneously. For example, let $\delta_3 \neq 0$. Then we express the vector f_3 in terms of the vectors f_0, f_1, f_2, substitute it into (1.10) and use (1.9). As a result we obtain a quadratic form in the quantities $(k, f_1), (k, f_2)$

$$L(k) = \delta_1^2 (k, f_1)^2 + (2\delta_1\delta_2 - \delta_3^2)(k, f_1), (k, f_2) + \delta_2^2 (k, f_2)^2 \tag{1.13}$$

which is non-negative if

$$\delta_1\delta_2 \geqslant \delta_3^2/4 \tag{1.14}$$

We can verify that the other cases ($\delta_1 \neq 0$ or $\delta_2 \neq 0$) also lead to inequality (1.14). Condition (1.14) is analogous to the conditions $D \geqslant 0, DC - B^2 \geqslant 0$.

Strict inequality in (1.14) ensures that form (1.13) is positive definite. In this case $L(k) = 0$ if and only if $(k, f_1) = (k, f_2) = 0$ under the condition $(k, f_0) = 0$. Because of the linear dependence of the vectors f_0, f_1, f_2, f_3 we also obtain $(k, f_3) = 0$. Note that the system of equations $(k, f_i) = 0$ $(i = 0, 1, 2, 3)$ always has non-trivial solutions when $n \geqslant 4$. From (1.8) it follows that $\mu_1 = \mu_2 = 0$, and the question of the optimality of the vector h_0 reduces to an investigation of the signs of the second variations of the double eigenvalue λ_0.

Consider the case $n = 3$. Let the rank of the matrix composed of the vectors f_0, f_1, f_2, f_3 equal three. Then the system of linear equations $(k, f_i) = 0$ $(i = 0, 1, 2, 3)$ admits of only the trivial solution $k = 0$. Thus, in this case the condition $\delta_1\delta_2 > \delta_3^2/4$ is a sufficient extremum condition in the optimization problem being considered, since $L(k) > 0$ for any non-zero k satisfying the condition $(k, f_0) = 0$. Similarly, in the case $n = 2$, when the rank of the matrix composed of the vectors f_0, f_1, f_2, f_3 equals two, the condition $\delta_1\delta_2 > \delta_3^2/4$ is a sufficient extrumum condition.

Example. Consider the oscillatory system shown in the figure /10/.

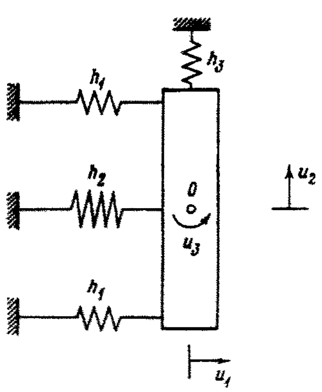

Fig.1

The oscillations of this system are described by the equation

$$\left\| \begin{matrix} \frac{2h_1 + h_2}{m} & 0 & 0 \\ 0 & \frac{h_3}{m} & 0 \\ 0 & 0 & \frac{2lh_1}{I} \end{matrix} \right\| \left\| \begin{matrix} u_1 \\ u_2 \\ u_3 \end{matrix} \right\| = \omega^2 \left\| \begin{matrix} u_1 \\ u_2 \\ u_3 \end{matrix} \right\|$$

where h_i $(i = 1, 2, 3)$ are the rigidities of the couplings, m is the mass of the rigid rod, $2l$ is its length, I is the rod's moment of inertia relative to the point O, and u_i $(i = 1, 2, 3)$ are the displacements and the angle of rotation. The eignvalues are

$$\lambda_1 = \omega_1^2 = \frac{2h_1 + h_2}{m}, \quad \lambda_2 = \omega_2^2 = \frac{h_3}{m}, \quad \lambda_3 = \omega_3^2 = \frac{2l}{I} h_1$$

respectively. If $2h_1 + h_2 = h_3$, then we have a double natural frequency (it is assumed that $2lh_1/I > h_3/m$).

We consider the problem of choosing the rigidities $h_i > 0$ $(i = 1, 2, 3)$ satisfying the condition

$$(c, h) = c_1h_1 + c_2h_2 + c_3h_3 = c_0 > 0 \tag{1.15}$$

and maximizing the double natural oscillation frequency. Here c_i $(i = 0, 1, 2, 3)$ are certain specified numbers.

In accordance with Eqs.(1.7) we write the vectors

$$f_1 = \frac{1}{m}(2, 1, 0), \quad f_2 = \frac{1}{m}(0, 0, 1), \quad f_3 = 0, \quad f_0 = (c_1, c_2, c_3)$$

From (1.10) and condition (1.9) become

$$L(k) = -(f_1, k)(f_2, k); \ (c, k) = 0; \ k = (k_1, k_2, k_3)$$

If the vectors f_0, f_1, f_2 are linearly independent, it is possible to construct a refining variation. Otherwise, we have the equation $c_1 = 2c_2$, being the first necessary optimality condition for the vector $(h_1, h_2, 2h_1 + h_2)$. Because of the linear dependence of the vectors we have $\delta_0 f_0 + \delta_1 f_1 + \delta_2 f_2 = 0$ with certain constants δ_0, δ_1 and δ_2. Hence we find $\delta_1 = -\delta_0 c_1 m/2$, $\delta_2 = -\delta_0 c_3 m$. Condition (1.14) becomes

$$\delta_1\delta_2 = \delta_0^2 m^2 c_1 c_3/2 \geqslant 0$$

Thus, the necessary extremum conditions reduce to: $c_1 = 2c_2$, $c_1c_3 \geqslant 0$. Using the isoperimetric condition and the condition for a minimum of the double natural frequency, we finally obtain the relation between the rigidities $h_i^\circ > 0$, which realizes the maximum of the minimum double eigenvalue

$$h_3^\circ = \frac{c_0}{c_2 + c_3}, \quad 2h_1^\circ + h_2^\circ = \frac{c_0}{c_2 + c_3}, \quad 2h_1^\circ > \frac{c_0}{c_2 + c_3}\frac{I}{lm}$$

where the parameters of the problem satisfy the inequalities

$$c_1 = 2c_2; \ c_i > 0, \ i = 0, 1, 2, 3; \ lm/I > 1$$

This example admits of an intuitive geometric interpretation of the necessary extremum conditions. In the case considered the vectors f_1, f_2, f_0 are vectors normal to the surfaces $\lambda_1 = \text{const}$, $\lambda_2 = \text{const}$ and to the constraint surface in the space of the parameters h_1, h_2, h_3. Therefore, the necessary extremum conditions reduce to the coplanarity of the vectors f_1, f_2, f_0 and to the condition that vector f_0 belong to the cone formed by f_1 and f_2.

We note that in view of the homogeneity of the functional $\min_i \lambda_i$ and the constraint (1.15) on the vector of variables $h = (h_1, h_2, h_3)$, the optimization problem being considered is equivalent to the dual problem /14/

$$\min_h (c, h) \quad \text{with} \quad \min_i \lambda_i = \lambda_0 = \text{const} > 0$$

2. The continuous case. Consider the problem of the loss of stability of an elastic rod of variable cross-section, compressed by a longitudinal force. In diemensionless variables the equations determining the buckling $w(x)$ when there is a loss of stability are /1,2/

$$(h^2 w'')'' = -\lambda w'', \ w(0) = w'(0) = w(1) = w'(1) = 0 \tag{2.1}$$

if it is assumed that the ends $x = 0$ and $x = 1$ the rod are clamped.

The optimization problem consists of determining the cross-sectional area $h(x)$ of the rod ($h(x)$ is a non-negative continuous function) for which the critical force of loss of stability λ_0 will be a maximum. It is assumed that the volume (weight) of the rod is fixed

$$\int_0^1 h(x)\,dx = 1 \tag{2.2}$$

Suppose that a function $h_0(x)$ satisfying (2.2) exists, for which the least of the eigenvalues of problem (2.1), λ_0, is double. Let $u_1(x)$ and $u_2(x)$ be two linearly independent functions corresponding to the multiple eigenvalue λ_0. Consider the linear subspace of all functions, formed from u_1 and u_2. In this subspace we choose the basis in a special way, namely, functions $w_1(x)$ and $w_2(x)$ such that (δ_{ij} is the Kronecker delta)

$$\int_0^1 w_i'(x) w_j'(x)\,dx = \delta_{ij}, \quad i, j = 1, 2 \tag{2.3}$$

Any eigenfunction corresponding to the double eigenvalue λ_0 can be represented as

$$w_0(x) = \gamma_1 w_1(x) + \gamma_2 w_2(x) \tag{2.4}$$

We give the function $h_0(x)$ an extension in the form $\varepsilon\delta h$, where ε is a small positive number. From (2.2) it follows that

$$\int_0^1 \delta h\,dx = 0 \tag{2.5}$$

Let us apply the results of an analytic perturbation of the spectrum of a selfadjoint operator /13/. We will denote by $v(x)$ a function representing an addition to the unperturbed eigenfunction $w_0(x)$. Arguing as in the finite-dimensional case, we arrive at the equations

$$(2h_0\delta h\, w_0'')'' + (h_0^2 v'')'' + \lambda_0 v'' + \mu w_0'' = 0$$
$$v(0) = v'(0) = 0, \quad v(1) = v'(1) = 0$$

Here μ denotes the magnitude of the addition to the multiple eigenvalue λ_0, which occurs when the function $h_0(x)$ is varied.

Scalarly multiplying the last equation by the functions $w_1(x)$ and $w_2(x)$ and using Eq.(2.1) with $\lambda = \lambda_0$ and $w = w_0(x)$, as well as the orthogonality property (2.3), we arrive at the equations determining the magnitude of the first correction with respect to ε of the perturbed eigenvalue, of the same as in Sect.1. For non-zero γ_1 and γ_2 to exist it is necessary that the condition

$$\mu^2 - (\beta_{11} + \beta_{22})\mu + \beta_{11}\beta_{22} - \beta_{12}^2 = 0 \tag{2.6}$$
$$\beta_{ij} = 2\int_0^1 h_0 w_i'' w_j'' \delta h\, dx, \quad i, j = 1, 2$$

be satisfied.

The symmetry of the matrix of the coefficients β_{ij} signifies that all roots of Eq.(2.6) are real. Just as in the discrete case, the solutions of Eq.(2.6) are invariant to the choice of the orthogonal basis w_1 and w_2 satisfying condition (2.3). The following statement holds.

3°. If the system of functions $f_1 = h_0(w_1'')^2$, $f_2 = h_0(w_2'')^2$, $f_3 = h_0 w_1'' w_2''$ and $f_0 = 1$ are linearly independent then a refining variation δh exists for which each of the perturbed eigenvalues will be larger than the original multiple eigenvalue λ_0.

Proof. Consider the functional form of the function δh

$$H(\delta h) = \frac{1}{4}(\beta_{12}^2 - \beta_{11}\beta_{22}) = \left[\int_0^1 h_0 w_1'' w_2'' \delta h\, dx\right]^2 - \left[\int_0^1 h_0 w_1''^2 \delta h\, dx\right]\left[\int_0^1 h_0 w_2''^2 \delta h\, dx\right] \tag{2.7}$$

As in the discrete case, we will show that when the functions f_0, f_1, f_2, f_3 are linearly independent a variation δh always exists satisfying the relations $\beta_{12} = 0$, $\beta_{11} = p_1 > 0$, $\beta_{22} = p_2 > 0$ and condition (2.5). We note that $\mu_1 = p_1 > 0$, $\mu_2 = p_2 > 0$ when these relations are satisfied. This signifies precisely that the variation δh is a refining variation.

Let us consider a linear functional in the space $L(0, 1)$ of absolutely integrable functions

$$F(f) = \int_0^1 f\delta h\, dx, \quad \int_0^1 |f|\, dx < \infty, \quad f \in L(0, 1) \tag{2.8}$$

where δh is a fixed bounded function.

The functions f_0, f_1, f_2, f_3 defined above belong to the space $L(0, 1)$ /15/. We consider the linear subspace $M \in L(0, 1)$ formed by the functions f_0, f_1, f_2, f_3. We will show that in the set M a linear functional (2.8) exists generated by some bounded function δh^* such that

$$\int_0^1 f_3 \delta h^*\, dx = 0, \quad \int_0^1 \delta h^*\, dx = 0, \quad \int_0^1 f_1 \delta h^*\, dx = p_1 > 0, \quad \int_0^1 f_2 \delta h^*\, dx = p_2 > 0$$

Indeed, for any function $f \in M$ the expansion $f = c_0 f_0(x) + \ldots c_3 f_3(x)$ holds. Hence follows the linearity of the functional

$$F^*(f) = \int_0^1 f\delta h^*\, dx, \quad f \in M$$

To prove the continuity of $F^*(f)$ it is sufficient to show that a number d exists for which

$$|F^*(f)| = \left|\int_0^1 \sum_{i=0}^3 c_i f_i \delta h^*\, dx\right| = |c_1 p_1 + c_2 p_2| \leqslant d\int_0^1 \left|\sum_{i=0}^3 c_i f_i\right| dx$$

Then

$$d = \sup_{\sum_{i=0}^3 c_i^2 = 1} |c_1 p_1 + c_2 p_2| \left(\int_0^1 |\Sigma c_i f_i|\, dx\right)^{-1}$$

The upper bound of the last expression exists on the strength of the assumption of the linear independence of the system of functions f_i $(i = 0, 1, 2, 3)$, which proves the continuity of $F^*(f)$.

The functional F^* constructed in this way in the set M can, by the Hahn-Banach theorem, be extended while preserving the norm (of the number d) over the whole space $L(0, 1)$ /16/. The latter signifies the possibility of constructing a refining variation in the case considered.

The following statement also holds.

4°. If the function h_0 provides an extremum of the optimization problem considered with a double eigenvalue λ_0, it is necessary that

1) the system of functions $f_0 = 1$, $f_1 = h_0 w_1''^2$, $f_2 = h_0 w_2''^2$, $f_3 = h_0 w_1'' w_2''$ should be linearly dependent, i.e., a constant δ_0, δ_1, δ_2 and δ_3 exists such that

$$[\delta_1 w_1''^2 + \delta_2 w_2''^2 + \delta_3 w_1'' w_2''] h_0 = \delta_0 \tag{2.9}$$

2) the functional form $H(\delta h)$ specified by Eq.(2.7) should admit of only non-negative values for any functions δh satisfying condition (2.5).

Proof. The necessity of the first condition follows from the preceding statement. Arguing as in the proof of statement 2°, we obtain the necessity of the second condition. We will simply show that form (2.7) can take zero values with admissible $\delta h \not\equiv 0$. For this we consider a linear subspace $M \subset L(0, 1)$ generated by the functions f_0, f_1, f_2, f_3 and in the space $L(0, 1)$ we choose an element $g(x)$ not belonging to the subspace M. We form a new subspace M' generated by the elements f_0, f_1, f_2, f_3 and g. Further, as in the proof of the preceding statement, let us show that a functional $F^*(f)$, $f \in M'$, exists, which takes zero values in the subspace M while $F^*(g) = 1$. While preserving the norm we extend the functional obtained, by the Hahn-Banach theorem, to the whole space $L(0, 1)$.

It is difficult to verify the condition of non-negativity of form (2.7) in the form presented. However, we can introduce a constructive condition suitable for practical use. It follows directly from condition (2.9) which can be written as $\delta_1 f_1 + \delta_2 f_2 + \delta_3 f_3 = \delta_0$. Arguing as in Sect.1, we obtain the condition for form (2.7) to be non-negative

$$\delta_1 \delta_2 \geqslant \delta_3^2/4$$

Example. Consider the problem of the loss of stability of a freely supported rod of variable cross-section lying on an elastic foundation. Introducing dimensionless variables, we write the equation for the rod's buckling

$$(h^2 u'')'' + ku + \lambda u'' = 0$$
$$u(0) = u(1) = 0, \quad h^2 u''(0) = h^2 u''(1) = 0$$

Here k is a quantity characterizing the coefficient of the elastic foundation's bed, and λ is an eigenvalue (the stability loss force).

It is well-known /17/ that when $h = h_0 \equiv 1$ problem (2.1) has a double eigenvalue when $k = 4\pi^4$. The corresponding eigenfunctions

$$u_1 = \frac{\sqrt{2}}{\pi} \sin \pi x, \quad u_2 = \frac{\sqrt{2}}{2\pi} \sin 2\pi x$$

are orthogonal and normalized in accordance with (2.3). We consider a variation δh satisfying the volume-constancy condition

$$\int_0^1 \delta h \, dx = 0 \tag{2.10}$$

The functions f_0, f_1, f_2, f_3 take the form

$$f_0 = 1, \; f_1 = 2\pi^2 \sin^2 \pi x = \pi^2 (1 - \cos 2\pi x)$$
$$f_2 = 8\pi^2 \sin^2 2\pi x = 4\pi^2 (1 - \cos 4\pi x)$$
$$f_3 = 4\pi^2 \sin \pi x \sin 2\pi x = 2\pi^2 (\cos \pi x - \cos 3\pi x)$$

These functions are linearly independent. Consequently, according to the statements proved it is possible to construct the refining variation.

Taking into account the isoperimetric condition (2.5) we write the form (2.7)

$$H(\delta h) = 4\pi^4 \left[\left(\int_0^1 (\cos \pi x - \cos 3\pi x) \, \delta h \, dx \right)^2 - \int_0^1 \cos 2\pi x \, \delta h \, dx \int_0^1 \cos 4\pi x \, \delta h \, dx \right]$$

We expand the function δh in a Fourier series in $\cos j\pi x$ $(j = 1, 2, 3 \ldots)$. The isoperimetric condition (2.10) is satisfied here for any coefficients c_j. Substituting the expansion of δh into form $H(\delta h)$, we obtain

$$H(\delta h) = \pi^4 [(c_1 - c_3)^2 - c_2 c_4]$$

As an example we take $c_1 = c_3 = 0$, $c_2 = c_4 = -1$. Then $H(\delta h) < 0$. In addition, in this case

$$\int_0^1 f_3 \, \delta h \, dx = 0$$

and consequently

$$\mu_1 = \int_0^1 f_1\, \delta h\, dx = \frac{\pi^2}{2}, \quad \mu_2 = \int_0^1 f_2\, \delta h\, dx = 2\pi^2$$

Thus, the refining variation has the form

$$\delta h(x) = -(\cos 2\pi x + \cos 4\pi x) + \sum_{j=5}^{\infty} c_j \cos j\pi x$$

where the coefficients $c_5, c_6, \ldots$ are arbitrary constants.

Note that if together with constraint (2.2) we impose additional conditions on the function h, then, using the arbitrariness of the coefficients $c_5, c_6, \ldots$, we can satisfy them.

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